MATHEMATICS METHODS

MAWA Semester 2 (Units 3 and 4) Examination 2016

Calculator-Assumed

Marking Key

The release date for this exam and marking scheme is

the end of week 1 of term 4, 2016

Section Two: Calculator-assumed (104 Marks)

Question 11

Solution

$y = 1 - e^{5x^2 - \ln(2-x)}$

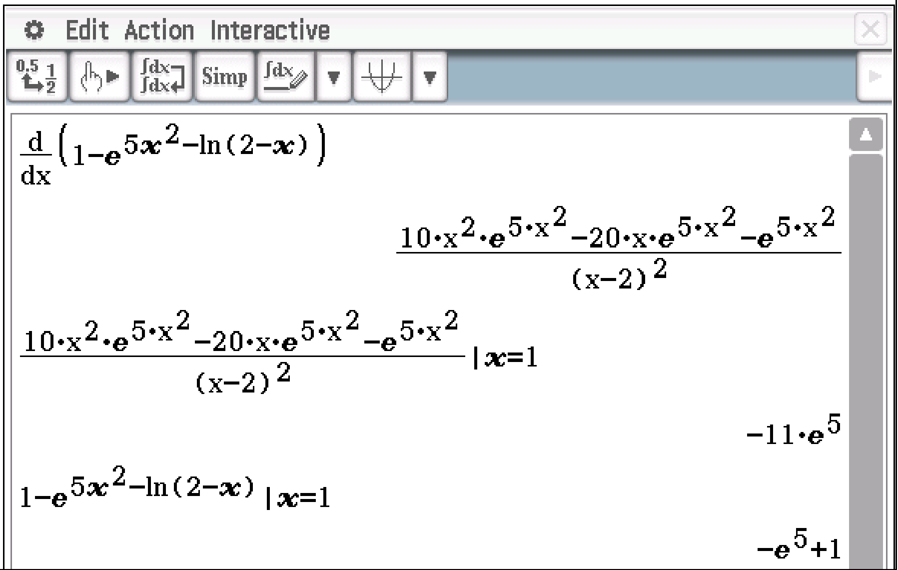
$\dfrac{dy}{dx} = -e^{5x^2 - \ln(2-x)} \times \left(10x + \dfrac{1}{2-x}\right)$

When $x = 1$,

$y = 1 - e^{5-\ln(1)} = 1 - e^5$ and $\dfrac{dy}{dx} = -e^{5-\ln(1)} \times (10+1) = -11e^5$

Equation of the tangent at $\left(1, 1-e^5\right)$ is

$y = -11e^5 x + 1 + 10e^5$



Marking key/mathematical behaviours	Marks
• determines derivative	1
• determines y and $\dfrac{dy}{dx}$ when $x = 1$	1
• states correct equation of tangent	1

Question 12(a)(i)

Solution

Total number of coins is $11 + k$

Seven 10 c coins, therefore P(10 c) = $\dfrac{7}{11+k}$ as required

Marking key/mathematical behaviours	Marks
• states correct total	1
• states correct probability	1

Question 12(a)(ii)

Solution

$$\frac{50}{11+k} + \frac{60}{11+k} + \frac{70}{11+k} + \frac{5k}{11+k} = 10$$

$$180 + 5k = 110 + 10k \quad \Rightarrow \quad k = 14$$

Marking key/mathematical behaviours	Marks
• states correct equation and equates equal to 10	1, 1
• calculates the value of k	1

Question 12(b)(i)

Solution

P(ignites at least once) ≥ 0.99

$\therefore$ P(no ignition) ≤ 0.01

i.e. $0.6^n \le 0.01 \quad \Rightarrow \quad n \ge 9.01$

Therefore, minimum number of trials is 10

Marking key/mathematical behaviours	Marks
• states first inequality	1
• correctly uses complementary events and solves for n	1
• correctly states minimum trials is 10	1

Question 12(b)(ii)

Solution

P(lighter uses last of fuel on 11^{th} trial) = ${}^{10}\mathbf{C}_8\,(0.4)^8(0.6)^2 \times 0.4 = 0.0042$

Marking key/mathematical behaviours	Marks
• correctly calculates probability of igniting 8 times in 10 trials	1
• multiplies by 0.4 (ignition on 11^{th} trial)	1
• calculates correct probability	1

Question 13(a)

Solution

$$\frac{dI}{dt} = rI$$
$$I = I_0 e^{rt}$$
$$2 = e^{12r}$$
$$r = \frac{\ln 2}{12} \; or \; \frac{100 \ln 2}{12}\%$$

Marking key/mathematical behaviours	Marks
• writes correct exponential equation for anti-derivative	1
• states $I = 2I_0$ or establishes this relationship in terms of money	1
• gives exact value of r	1

Question 13(b)

Solution

$$\frac{dS}{dx} = \frac{b}{5x+2} = \frac{b}{5} \times \frac{5}{5x+2}$$
$$S = \frac{b}{5}\ln(5x+2) + c$$
$$x = 0, S = 65 \Rightarrow 65 = \frac{b}{5}\ln(2) + c \Rightarrow c = 65 - \frac{b}{5}\ln(2)$$
$$S = \frac{b}{5}\ln(5x+2) + 65 - \frac{b}{5}\ln(2)$$
$$= \frac{b}{5}\left[\ln(5x+2) - \ln(2)\right] + 65$$
$$= \frac{b}{5}\left[\ln\left(\frac{5x+2}{2}\right)\right] + 65$$

Marking key/mathematical behaviours	Marks
• determines correct anti-derivative of function plus c	1
• calculates value of constant term	1
• writes expression for S	1
• factorises correctly	1
• correctly uses log law and deduces correct expression for S	1

Question 14(a)

Solution	
Let point P have coords: (a,b) $g(x)=mx+c$ $\therefore g(x)=-6\cos(3a)x+c$ Since $g(0)=0$: $c=0, \quad \therefore g(x)=-6\cos(3a)x$ since $g(a)=b$ $b=-6\cos(3a)(a)$.....(1) Also since $f(a)=b$ $b=-2\sin(3a)$..........(2) Solving eqn (1) and (2) gives $a\approx 1.4978\approx 1.50$ $b\approx 1.9522\approx 1.95$	
Marking key/mathematical behaviours	**Marks**
• determines equation for $g(x)$ involving a. • states equation (1) • states equation (2) • solves for a and b.	1 1 1 1

Question 14(b)

Solution	
$g(x)=-6\cos(3(1.5))x$ $g(x)=1.3x$ OR $m=\dfrac{1.95}{1.5}=1.3$ $g(x)=1.3x$	
Marking key/mathematical behaviours	**Marks**
• substitutes value a = 1.5 or determines gradient • writes equation	1 1

Question 14(c)

Solution	
$Area=\int_0^{1.5}\{g(x)-f(x)\}dx$ $=\int_0^{1.5}\{1.3x-(-2\sin(3x))\}dx$ ≈ 2.27 square units	
Marking key/mathematical behaviours	**Marks**
• uses boundaries of 0 and 1.5 • writes an appropriate integral representing the required area • calculates area	1 1 1

Question 15(a)

Solution	
$f(x) = x\ln x - x + 3$ $f'(x) = 1\times\ln x + x\times\frac{1}{x} - 1$ $= \ln x + 1 - 1$ $= \ln x$	
Marking key/mathematical behaviours	Marks
• shows process to determine correct expression	1

Question 15(b)

Solution	
$\int \ln x dx = x\ln x - x + c$	
Marking key/mathematical behaviours	Marks
• recognises the integral is $f(x)$ from part (i) but with an unknown constant	1

Question 15(c)

Solution	
$g(x) = \int \ln(x^2)dx = \int 2\ln xdx = 2\int \ln xdx = 2(x\ln x - x + c) = 2x\ln x - 2x + k$	
Marking key/mathematical behaviours	Marks
• uses relationship $\ln(x^n) = n\ln x$ • substitutes correct expression for the integral and simplifies correctly	1 1

Question 16

Solution

$$v(t) = \int (kt - 6)\,dt$$
$$= \frac{kt^2}{2} - 6t + c$$

Since $v(0) = 3,\ k = 0 \Rightarrow v(t) = \dfrac{kt^2}{2} - 6t + 3$

$$x(t) = \int \left(\frac{kt^2}{2} - 6t + 3\right) dt$$
$$x(t) = \frac{kt^3}{6} - 3t^2 + 3t + c$$

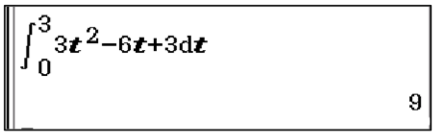
$$x(2) = 19 \Rightarrow 19 = \frac{8k}{6} - 6 + c \ldots\ldots(1)$$
$$x(3) = 26 \Rightarrow 26 = \frac{9k}{2} - 18 + c \ldots\ldots(2)$$

Solving (1) and (2) gives:

$k = 6$ and $c = 17$

$v(t) = 3t^2 - 6t + 3$

dist travelled $= \displaystyle\int_0^3 3t^2 - 6t + 3\,dt = 9\,\text{m}$



Marking key/mathematical behaviours	Marks
• integrates to determine $v(t)$	1
• integrates to determine $x(t)$ with constant	1
• writes equations 1 and 2	1
• solves for c and k	1
• writes integral to calculate distance travelled	1
• calculates distance travelled.	1

Question 17(a)

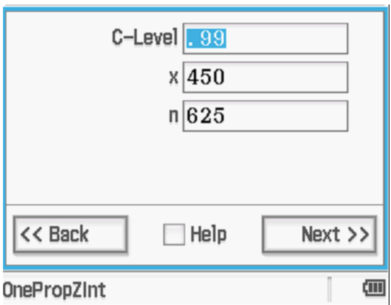
Solution

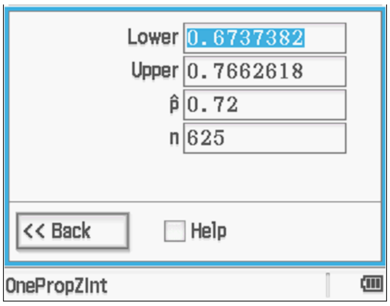
$$\hat{p} = \frac{450}{625} = 0.72$$

Marking key/mathematical behaviours	Marks
• Calculates proportion	1

Question 17(b)

Solution





Hence $0.674 \le p \le 0.766$

We can be 99% confident that that between 67.4% and 76.6% of ABC customers used online banking to pay their bills.

Marking key/mathematical behaviours	Marks
• correctly calculates lower value of confidence interval • correctly calculates upper value of confidence interval • interprets answer correctly	1 1 1

Question 17(c)

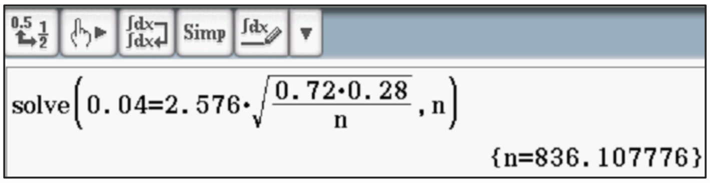
Solution

$$E = 2.576 \times \sqrt{\frac{0.72 \times 0.28}{n}}$$

$$0.02 = 2.576 \times \sqrt{\frac{0.72 \times 0.28}{n}}$$

$$n = 836.108$$

$$n \approx 837$$



Marking key/mathematical behaviours	Marks
• states standard error • writes an equation to evaluate n • solves correctly for n • rounds n up to the nearest integer.	1 1 1 1

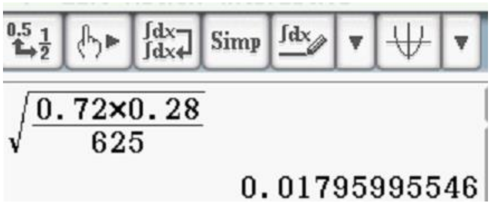
Question 17(d)

Solution

We would expect the sample proportions to be approximately normally distributed.

$\text{Mean} = p \approx 0.72$

$\text{Standard deviation} \approx \sqrt{\dfrac{0.72 \times 0.28}{625}} = 0.018$



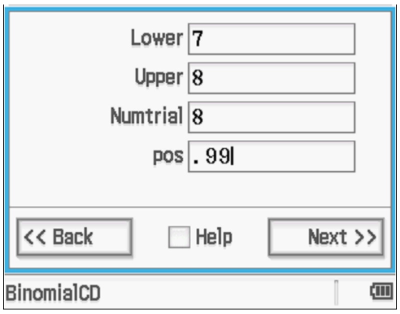
Marking key/mathematical behaviours	Marks
• states distribution	1
• calculates the mean	1
• calculates the Standard deviation	1

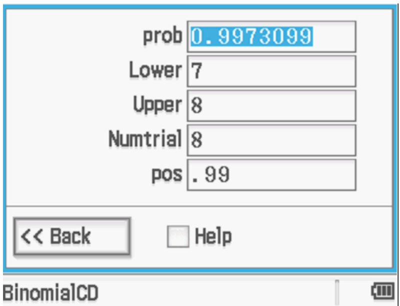
Question 17(e)

Solution

Binomial Distribution

$X \sim bin(8, 0.99) \Rightarrow P(x \geq 7) = 0.9973$





Marking key/mathematical behaviours	Marks
• identifies distribution as binomial including parameters	1
• calculates probability	1

Question 18(a)(i)

Solution

Intensity of the sound of a vacuum cleaner, I_V

$$70 = 10\log\left(\frac{I_V}{I_0}\right) \Rightarrow 7 = \log\left(\frac{I_V}{I_0}\right) \Rightarrow 10^7 = \frac{I_V}{I_0} \Rightarrow 10^7 \times I_0 = I_V$$

Marking key/mathematical behaviours	Marks
• arrives at correct expression for I_V	1

Question 18(a)(ii)

Solution

Intensity of the sound of an electric drill, $I_D = 10^{9.8} \times I_0$

$$\frac{I_D}{I_V} = \frac{10^{9.8} \times I_0}{10^7 \times I_0} = 10^{2.8} = 631$$

So the intensity of the sound of an electric drill is 631 times greater than the intensity for the sound of a vacuum cleaner.

Marking key/mathematical behaviours	Marks
• compares intensities of the 2 sounds	1
• states correct relationship	1

Question 18(a)(iii)

Solution

$$L = 10\log\left(\frac{10^{9.8} \times I_0}{I_0}\right) = 10\log 10^{9.8} = 10 \times 9.8 \log 10 = 98 \text{ decibels}$$

Marking key/mathematical behaviours	Marks
• calculates correct value	1

Question 18(b)(i)

Solution

acceleration $= \dfrac{dv}{dt} = -2\sin 2t$

Marking key/mathematical behaviours	Marks
• differentiates correctly	1

Question 18(b)(ii)

Solution

When $t = \pi, v = \cos 2\pi = 1$

and $\dfrac{dv}{dt} = -2\sin 2\pi = 0$

Marking key/mathematical behaviours	Marks
• determines correct value for v and for $\dfrac{dv}{dt}$	1, 1

Question 18(b)(iii)

Solution

$\dfrac{dv}{dt} = 0$ indicates that $t = \pi$ gives a local maximum or minimum value for v .The maximum value of function $v = cos2t$ is 1, so the particle is travelling at its maximum velocity at $t = \pi$.

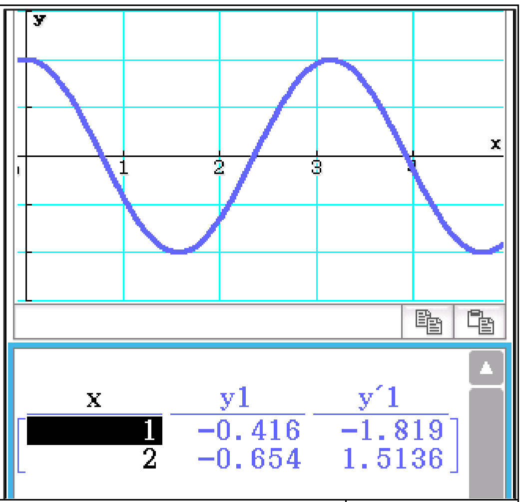
Marking key/mathematical behaviours	Marks
• identifies significance of rate of change = 0	1

Question 18(b)(iv)

Solution

The graph shows $v = \cos 2t$. It can be seen that the gradient (which is acceleration) is negative before the minimum and positive after the minimum. This can also be seen from the table.

During this particular second, the velocity is decreasing until it reaches its minimum value and then the velocity increases.



Marking key/mathematical behaviours	Marks
• gives a correct interpretation of the given facts	1

Question 19(a)

Solution

$$k\int_1^2 x^2 dx = 1 \quad \Rightarrow \quad \frac{k}{3}\left[x^3\right]_1^2 \quad \Rightarrow \quad \frac{7k}{3} = 1 \therefore k = \frac{3}{7}$$

Marking key/mathematical behaviours	Marks
• equates integral equal to one	1
• determines the value of k	1

Question 19(b)

Solution

$$E(X) = \int_1^2 x \ \frac{3x^2}{7} dx = \frac{45}{28}$$

$$E(X^2) = \int_1^2 x^2 \ \frac{3x^2}{7} dx = \frac{93}{35}$$

$$Var(X) = \frac{93}{35} - \left(\frac{45}{28}\right)^2 = \frac{291}{3920} \quad \Rightarrow \quad \sigma = 0.272$$

Marking key/mathematical behaviours	Marks
• correctly calculates E(X)	1
• correctly calculates E(X^2)	1
• correctly calculates Var(X), hence standard deviation	1

Question 19(c)

Solution

$$F(x) = \int_1^k \frac{3x}{7} dx = \left[\frac{x^3}{7}\right]_1^k = \frac{k^3}{7} - \frac{1}{7}$$

$$\therefore F(x) = \begin{cases} 0 & x < 1 \\ \dfrac{x^3}{7} - \dfrac{1}{7} & 1 \leq x \leq 2 \\ 1 & x > 2 \end{cases}$$

Marking key/mathematical behaviours	Marks
• correctly sets up integral for F(x)	1
• correctly writes F(x) as a piecewise function	1
• uses correct boundaries	1

Question 19(d)

Solution

$$\int_1^m \frac{3x^2}{7}\,dx = 0.5 \;\Rightarrow\; m = \sqrt[3]{\frac{9}{2}} = 1.65$$

Marking key/mathematical behaviours	Marks
• states correct integral	1
• calculates the value of m	1

Question 20(a)

Solution

$$Yr7 = \frac{305}{1032}\times 75 \approx 22$$

$$Yr8 = \frac{381}{1032}\times 75 \approx 28$$

$$Yr7 = \frac{346}{1032}\times 75 \approx 25$$

Marking key/mathematical behaviours	Marks
• determines proportions	1
• dives integer values for each year group	1

Question 20(b)(i)(ii)

Solution

(i) Uniform Distribution

(ii)
$$\begin{aligned} mean &= E(X) \\ &= \frac{6+1}{2} \\ &= 3.5 \end{aligned}$$

Marking key/mathematical behaviours	Marks
• states distribution	1
• states mean	1

Question 20(b)(iii)

Solution

The bars would be higher but have much less variation in height

Marking key/mathematical behaviours	Marks
• states reasons	1

Question 21 (a)

Solution Sample 3. The largest sample size is likely to give the best estimate.	
Marking key/mathematical behaviours	Marks
• identifies correct sample with reason	1

Question 21 (b)

Solution $p \approx 0.021413$ No allowance is made for the sample size. The larger sample sizes are likely to give more reliable estimates.	
Marking key/mathematical behaviours	Marks
• states the approximation of p • discusses sample size as a factor in reliability	1 1

Question 21 (c)

Solution

Best method would be to calculate the total defective items from all the samples and use the sum of all the sample sizes to determine the proportion estimate. This effectively makes a very large sample size and hence gives the best estimate.

Sample	1	2	3	4	5	6	7	8	total
Number in sample	122	72	450	158	280	150	205	310	1747
Number of Defective components	3	1	10	4	7	2	5	7	39

$p \approx \frac{39}{1747} = 0.0223$

Marking key/mathematical behaviours	Marks
• states reason for better estimate • determines number of defective components in each sample • calculates total sample size and total defective components • calculates mean to estimate the population proportion	1 1 1 1

Question 22 (a)

Solution	
$P(X > 6.54) = \frac{1}{15} = 0.0667$ $P\left(Z > \frac{6.54 - 6.50}{\sigma}\right) = 0.0667 \Rightarrow \frac{0.04}{\sigma} = 1.501 \therefore \sigma = 0.266$ using CAS	
Marking key/mathematical behaviours	**Marks**
• uses correct probability • correctly converts to z score • correctly solves for σ	1 1 1

Question 22(b)

Solution	
$P(X > 6.54) = \frac{1}{20} = 0.05$ $P\left(Z > \frac{6.54 - 6.50}{\sigma_1}\right) = 0.05 \Rightarrow \frac{0.04}{\sigma_1} = 1.645 \therefore \sigma_1 = 0.0243$ using CAS (σ_1 is the original standard deviation) Let the new mean be μ $P\left(Z > \frac{6.54 - \mu}{0.0243}\right) = 0.0667 \Rightarrow \frac{6.54 - \mu}{0.0243} = 1.501 \Rightarrow \mu = 6.504 \text{ cm}$	
Marking key/mathematical behaviours	**Marks**
• uses correct probability to calculate original std deviation • determines standard deviation using CAS • uses correct probability to calculate new mean • determines new mean using CAS	1 1 1 1

Question 22(c)

Solution	
$P(6.48 < X < 6.53) = 0.6442$ (where $X \sim N(6.50, 0.0266^2)$) Therefore, would expect 0.6442(1000) = 644 to have lengths in the required range	
Marking key/mathematical behaviours	**Marks**
• calculates probability and correct number of components	1

Question 23(a)

Solution

$$E = \frac{\frac{d}{dx}(\ln f(x))}{\frac{d}{dx}(\ln x)} = \frac{\frac{f'(x)}{f(x)}}{\frac{1}{x}} = \left(\frac{f'(x)}{f(x)}\right)\left(\frac{x}{1}\right) = \left(\frac{dq}{dx}\right)\left(\frac{x}{q}\right)$$, since $f'(x) = \frac{dq}{dx}$ and $f(x) = q$

Marking key/mathematical behaviours	Marks
• correctly differentiates numerator and denominator	1
• simplifies correctly	1
• gives reasons for simplification	1

Question 23(b)

Solution

Using $\left(\frac{dq}{dx}\right)\left(\frac{x}{q}\right) = (\alpha k x^{\alpha-1})\frac{x}{kx^{\alpha}} = \alpha$

Marking key/mathematical behaviours	Marks
• correctly differentiates using $\left(\frac{dq}{dx}\right)\left(\frac{x}{q}\right)$	1
• simplifies correctly	1

Question 23(c)

Solution

$$y = \ln(1+\sin x)$$

$$y' = \frac{\cos x}{1+\sin x}$$

$$y'' = \frac{(1+\sin x)(-\sin x) - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$= -\left(\frac{1+\sin x}{(1+\sin x)^2}\right)$$

$$= -\frac{1}{1+\sin x} = \frac{1}{e^y} \Rightarrow \frac{d^2y}{dx^2} + e^{-y} = 0$$

Marking key/mathematical behaviours	Marks
• determines first and second derivative	1
• simplifies correctly	1
• substitutes $1+\sin x = e^y$	1
• concludes proof	1

Question 23(d)

Solution	
$\frac{d^2y}{dx^2} = -e^{-y} < 0 \; \forall \; y \therefore$ all stationary points are maxima	
Marking key/mathematical behaviours	**Marks**
• states second derivative is negative	1
• concludes stationary points are maxima	1

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